Representations of Lie Algebras

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ΟΧΙ ΑΝΤΙΜΕΤΑΘΕΤΙΚΗ ΑΛΓΕΒΡΑ

Representations of Lie Algebras = Example of module theory on an non commutative and non associative algebra

ΟΧΙ ΠΡΟΣΑΙΤΕΡΙΣΤΙΚΗ ΑΛΓ.

Lie Algebras Representations- Bibliography

- J. E. Humphreys *Introduction to Lie Algebras and Representation Theory*,
  Graduate Texts in Mathematics, Springer 1980
- Alex. Kirillov *An Introduction to Lie Groups and Lie Algebras*,
  Cambridge Univ. Press 2008
- B. C. Hall *Lie Groups, Lie Algebras and Representations*,
  Grad. Texts in Maths. Springer 2003
- R. W. CARTER *Lie Algebras of Finite and Affine Type*,
  Cambridge Univ. Press 2005
- K. Erdmann and M. J. Wildon *Introduction to Lie Algebras*, Spinger 2006
- Hans Samelson, *Notes on Lie Algebras*
- Anne Moreau *Algèbres de Lie semisimples: reprèsentations et èlements nilpotents*,
  math.univ-lille1.fr/amoreau
- Andreas Čap, *Lie Algebras and Representation Theory*,
  Univ. Wien, Lecture Notes 2003
- Alberto Elduque, *Lie algebras*,
  Univ. Zaragoza, Lecture Notes 2005

$\mathbb{F}$ = Field of characteristic 0 (ex $\mathbb{R}$, $\mathbb{C}$),
$\alpha, \beta, \ldots \in \mathbb{F}$ and $x, y, \ldots \in \mathcal{A}$

Εδώ θα θεωρούμε ότι το $\mathbb{F} = \mathbb{C}$

Definition of an Algebra $\mathcal{A}$

$\mathcal{A}$ is a $\mathbb{F}$-vector space with an additional distributive binary operation or product

$$\mathcal{A} \times \mathcal{A} \ni (x, y) \xrightarrow{m} m(x, y) = x * y \in \mathcal{A}$$

- $\mathbb{F}$-vector space
  $\alpha x = x\alpha, \; (\alpha + \beta)x = \alpha x + \beta x, \; \alpha(x + y) = \alpha x + \alpha y$
- distributivity
  $(x + y) * z = x * z + y * z, \; x * (y + z) = x * y + x * z$
  $\alpha(x * y) = (\alpha x) * y = x * (\alpha y) = (x * y)\alpha$

associative algebra $\equiv (x * y) * z = x * (y * z)$,
NON associative algebra $\equiv (x * y) * z \neq x * (y * z)$, ex. Lie Algebra
commutative algebra $\equiv x * y = y * x$,
NON commutative algebra $\equiv x * y \neq y * x$, ex. Matrices, Lie Algebra

Examples

Associative Algebra $\equiv (x * y) * z = x * (y * z)$

Example

The $n \times n$ complex (or real) Matrices $M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$)

$$A,\ B,\ C \text{ matrices} \Longrightarrow A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

NON Associative Algebra $\equiv (x * y) * z \neq x * (y * z)$

Example

$$\mathbf{a},\ \mathbf{b},\ \mathbf{c} \text{ vectors in } \mathbb{R}^3 \Longrightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

commutative algebra $\equiv x * y = y * x$ π.χ. Το σύνολο των πολυωνύμων

Example

$$a, b \in \mathbb{C} \Longrightarrow ab = ba$$

NON commutative algebra $\equiv x * y \neq y * x$, ex. Matrices, Lie Algebra

$$A,\ B,\ C \text{ matrices} \Longrightarrow A \cdot B \neq B \cdot A$$

Definition Lie Algebra

$\mathbb{F}$ = Field of characteristic 0 (ex $\mathbb{R}$, $\mathbb{C}$),
$\mathfrak{g}$ = $\mathbb{F}$-Algebra with a product or **bracket** or **commutator**
$\alpha, \beta, \ldots \in \mathbb{F}$ and $x, y, \ldots \in \mathfrak{g}$

⟷ (Αντι)μεταθέτης

$$\mathfrak{g} \times \mathfrak{g} \ni (x, y) \longrightarrow [x, y] \in \mathfrak{g}$$

Definition: Lie algebra Axioms

(Lie-i) **bi-linearity**:
$$[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]$$
$$[x, \alpha y + \beta z] = \alpha [x, y] + \beta [x, z]$$

(Lie-ii) **anti-commutativity**: $[x, y] = -[y, x] \;\Leftrightarrow\; [x, x] = 0$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 1}}$

(Lie-iii) **Jacobi identity**: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

or **Leibnitz rule**: $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

Αναλογία με τον κανόνα του Leibnitz στις συναρτήσεις

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$\Rightarrow$ Η θεωρία των Lie αλγεβρών συνδέεται με τις "παραγωγίσεις"

ΣΗΜΕΙΩΣΗ 1

$$\underset{(L)}{[x,y]=-[y,x]} \iff \underset{(R)}{[z,z]=0}$$

Απόδειξη $(R) \Rightarrow (L)$

$$\underbrace{[x+y,\,x+y]}_{=0} \overset{\text{Bilinearity}}{=} \underbrace{[x,x]}_{=0}+[x,y]+[y,x]+\underbrace{[y,y]}_{=0} \Rightarrow [x,y]+[y,x]=0$$

Απόδειξη $(L) \Rightarrow (R)$

Αν $[x,y]+[y,x]=0 \Rightarrow$ Όταν $y=x \rightsquigarrow 2[x,x]=0 \rightsquigarrow [x,x]=0$

Examples of Lie algebras

(i) $\mathfrak{g} = \{\vec{x}, \vec{y}, \ldots\}$ real vector space $\mathbb{R}^3$ with the commutator $[\vec{x}, \vec{y}] \stackrel{\text{def}}{\equiv} \vec{x} \times \vec{y}$ is a Lie algebra. ΑΣΚΗΣΗ 1

(ii) $\mathcal{A}$ associative $\mathbb{F}$- algebra $\rightsquigarrow$ $\mathcal{A}$ a Lie algebra $L(\mathcal{A})$ with commutator $[A, B] \stackrel{\text{def}}{\equiv} AB - BA$ ΣΗΜΕΙΩΣΗ 2

(iii) Angular Momentum in Quantum Mechanics ΑΣΚΗΣΗ 2

$$L_1,\, L_2,\, L_3 \quad \left\{ \begin{array}{c} \text{auto-adjoint linear differential} \\ \text{operators on } C^2(\mathbb{R}^3) \text{ functions} \end{array} \right\} f(x, y, z)$$

$$L_1 = i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),\; L_2 = i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),\; L_3 = i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\mathfrak{g} = \operatorname{span}\{L_1, L_2, L_3\}$$

$$[L_i, L_j] \equiv L_i \circ L_j - L_j \circ L_i$$

$$\rightsquigarrow [L_1, L_2] = iL_3, \quad [L_2, L_3] = iL_1, \quad [L_3, L_1] = iL_2$$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 2}}$ $\quad A, B, C \in \mathcal{A}$

$\mathcal{A}$ προσαιτεριστική άλγεβρα δηλ $(AB)C = A(BC)$

$L(\mathcal{A})$ = άλγεβρα $\mathcal{A}$ με ένα commutator

$$[A,B] \stackrel{def}{=} AB - BA.$$

Το $[A,B]$ είναι διγραμμικό και μεταθετικό και ικανοποιεί την ταυτότητα Jacobi

Απόδειξη:

$$[A,[B,C]] = A[B,C] - [B,C]A = A(BC) - A(CB) - (BC)A + (CB)A$$

$$[B,[C,A]] = B(CA) - B(AC) - (CA)B + (AC)B$$ ← προκύπτει από το προηγούμενη σχέση αν $A \to B$, $B \to C$, $C \to A$

$$[C,[A,B]] = C(AB) - C(BA) - (AB)C + (BA)C$$

Η προσαιτεριστικότητα της $\mathcal{A}$ σημαίνει ότι μπορούμε να αγνοήσουμε τις παρενθέσεις δηλδ $(AB)C = A(BC) = ABC$

οπότε $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

$\boxed{\text{ΟΠΟΤΕ Η } L(\mathcal{A}) \text{ είναι Lie ΑΛΓΕΒΡΑ}}$

ΕΦΑΡΜΟΓΗ

Αν V γραμμικός χώρος και $End(V)$ το σύνολο των ενδομορφισμών του V δηλ. Αν $f \in End(V) \rightarrow f : V \longrightarrow V$, f γραμμική απεικόνιση. Το $End(V)$ είναι μιά άλγεβρα με γινόμενο το $f \circ g$ δηλ. την σύνθεση των f και g, και το $End(V)$ είναι προσαιτεριστική άλγεβρα.

Τότε το $L(End(V)) \overset{def}{\equiv} gl(V)$ είναι Lie άλγεβρα με commutator το

$$[f, g] \overset{def}{\equiv} f \circ g - g \circ f$$

(iv) The quaternion algebra $\mathbb{H}(\mathbb{C})$ is an associative $\mathbb{C}$-algebra with the basis $\{e, i, j, k\}$ and multiplication table:

ΑΣΚΗΣΗ 3

	e	i	j	k
e	e	i	j	k
i	i	$-e$	k	$-j$
j	j	$-k$	$-e$	i
k	k	j	$-i$	$-e$

The quaternion **algebra** is a Lie algebra by introducing the commutator: $[A,\, B] = AB - BA$, where $A,\, B \in \mathbb{H}(\mathbb{C})$

$$A = A_0 e + A_1 i + A_2 j + A_3 k = A_0 e + \overline{A} \cdot \overline{\tau},$$
$$\overline{A} = (A_1,\, A_2,\, A_3)\,, \quad \overline{\tau} = (\tau_1,\, \tau_2,\, \tau_3) = (i, j, k)$$

$$AB = \left(A_0 B_0 - \overline{A} \cdot \overline{B}\right) e + \left(A_0 \overline{B} + B_0 \overline{A} + \overline{A} \times \overline{B}\right) \cdot \overline{\tau}$$

$$[A,\, B] = 2\overline{A} \times \overline{B} \cdot \overline{\tau}$$

Definition $\mathfrak{sl}(2,\mathbb{C})$ Algebra

$\mathfrak{sl}(2,\mathbb{C}) = \mathrm{span}\,(h,\ x,\ y)$

$$[h,\,x] = 2x \qquad [h,\,y] = -2y \qquad [x,\,y] = h$$

Examples

Ex. 1: 2×2 matrices with trace 0

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Ex. 2: Derivative operators acting on polynomials of order n

$\boxed{\text{ΑΣΚΗΣΗ 4}}$

$$h = 2z\frac{\partial}{\partial z} - n, \quad x = \underbrace{-z^2\frac{\partial}{\partial z} + nz}_{\text{διορθωμένο } x}, \quad y = \frac{\partial}{\partial z}$$

(vi) A **Poisson algebra** is a vector space over a field $\mathbb{F}$ equipped with two bilinear products, $\cdot$ and $\{\,,\,\}$, having the following properties:

(a) The product $\cdot$ forms an associative (commutative) $\mathbb{F}$-algebra.

(b) The product $\{\,,\,\}$, called the **Poisson bracket**, forms a Lie algebra, and so it is anti-symmetric, and obeys the Jacobi identity.

(c) The Poisson bracket acts as a derivation of the associative product $\cdot$, so that for any three elements x, y and z in the algebra, one has $\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$

Example: **Poisson algebra on a Poisson manifold**

M is a manifold, $C^\infty(M)$ the "smooth"/analytic (complex) functions on the manifold.

$$\{f, g\} = \sum_{ij} \omega_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

$$\omega_{ij}(x) = -\omega_{ji}(x), \quad \sum_m \left(\omega_{km} \frac{\partial \omega_{ij}}{\partial x_m} + \omega_{im} \frac{\partial \omega_{jk}}{\partial x_m} + \omega_{jm} \frac{\partial \omega_{ki}}{\partial x_m} \right) = 0$$